Initial Time Difference Strict Stability Criteria of Fractional Order Differential Equations in Caputo's Sense ⋆

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Abstract: We have investigated that initial time difference fractional strict stability criteria for unperturbed fractional differential systems with Caputo's Derivative. We establish comparison results for unperturbed fractional differential systems with respect to another unperturbed fractional differential systems which have different initial position and initial time.

Keywords: Initial time difference, strict stability, fractional order differential equation, Lyapunov's stability.

1. INTRODUCTION

The theory of fractional differential equations has been realized that it has very useful models for studying and understanding the various disciplines and processes of engineering applications. The development and improvement of this theory are very important for math and other areas of modern science. The history of fractional order derivative and fractional order differential equations goes back to the 17th century. It had always attracted the interest of many famous mathematicians, including L' Hospital, Leibnitz, Liouville, Riemann, Grünwald and Letnikov. Please See Kilbas et al. (2006), Lakshmikantham et al. (2009), Podlubny (1999) and Samko et al. (1993). In recent decades, fractional order differential equations have been found to be a powerful tool some fields, such as physics, mechanics, and engineering and it was realized that the derivatives of non-integer order provide an perfect framework for modelling of the real world applications in related disciplines from physics, chemistry and engineering [Kilbas et al. (2006) and Samko et al. (1993)].

The strict stability criteria had been studied by Lakshmikantham et al. (2001) and Yakar (2007). We have investigated that the strict stability criteria between two unperturbed differential systems with different initial time and initial position of fractional order with initial time difference . The differential operators are taken in the Riemann-Liouville and Caputo's sense and the initial conditions are specified according to Caputo's suggestion [Caputo (1967)], thus allowing for interpretation in a physically meaningful way [Kilbas et al. (2006), Lakshmikantham et al. (2009), Podlubny (1999) and Samko et al. (1993)].

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The initial time difference stability is very important for dynamic systems. It has been worked by Lakshmikantham and Vatsala [Lakshmikantham et al. (2009)] and Yakar [Shaw et al. (1999, 2000), Yakar (2010), Yakar (2007), Yakar et al. (2005), Yakar et al. (2008) and Yakar et al. (2009)]. We develop initial time difference fractional strict stability criteria for unperturbed fractional order differential systems with Caputo's derivative. We establish comparison results for unperturbed fractional order differential systems with respect to another unperturbed fractional order differential systems which have different initial position and initial time. The difference of these systems is that they have different initial conditions. The Lyapunov stability is with respect to null solution. The difference of these definitions and results from Lyapunov stability [Lakshmikantham et al. (1989), Lakshmikantham et al. (2009) and Lakshmikantham et al. (2001)] is that these systems stability investigates with respect to another unperturbed fractional differential systems which have different initial position and initial time.

2. DEFINITION AND NOTATION

The definition of Caputo' s and Reimann-Liouville's fractional derivatives

$$^{c}D^{q}x = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x'(s)ds, t_0 \le t \le T \quad (2.1)$$

$$D^{q}x = \frac{1}{\Gamma(p)}\left(\frac{d}{dt}\int_{\tau_0}^{t}(t-s)^{p-1}x(s)ds\right), t_0 \le t \le T \quad (2.2)$$

order of $0 < q < 1$, and $p + q = 1$ where Γ denotes the Gamma function.

The main advantage of Caputo's approach is that the initial conditions for fractional order differential equations with Caputo derivative take on the same form as that of ordinary differential equations with integer derivatives and another difference is that the Caputo derivative for a constant C is zero, while the Riemann-Liouville fractional derivative for a constant C is not zero but equals to $D^q C = \frac{C(t-\tau_0)^{-q}}{\Gamma(1-q)}$. By using (2.1) and therefore,

$$^cD^q x(t) = D^q\left[x(t) - x(t_0)\right] \tag{2.3}$$

$$^cD^q x(t) = D^q x(t) - \frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q} \tag{2.4}$$

In particular, if $x(t_0) = 0$, we obtain

$$^cD^q x(t) = D^q x(t). \tag{2.5}$$

Hence, we can see that Caputo' s derivative is defined for functions for which Riemann-Liouville fractional order derivative exists.

Consider the initial value problems of the fractional order differential equations with Caputo's fractional derivative

$$^cD^q x = f(t,x),\ x(t_0) = x_0 \text{ for } t \geq t_0,\ t_0 \in \mathbb{R}_+ \tag{2.6}$$

$$^cD^q y = f(t,y),\ y(\tau_0) = y_0 \text{ for } t \geq \tau_0 \geq t_0 \tag{2.7}$$

where $x_0 = \lim_{t\to t_0} D^{q-1}x(t)$ and $y_0 = \lim_{t\to\tau_0} D^{q-1}y(t)$ exist and $f \in C[[t_0, \tau_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$; satisfy a local Lipschitz condition on the set $\mathbb{R}_+ \times S\rho$, $S\rho = [x \in \mathbb{R}^n : \|x\| \leq \rho < \infty]$ and $f(t,0) = 0$ for $t \geq 0$.

We assume that we have sufficient conditions to the existence and uniqueness of solutions through (t_0, x_0) and (τ_0, y_0). If $f \in C[[t_0, \tau_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$ and $x(t)$ is the solution of the system (2.6) where $^cD^q x$ is the Caputo fractional order derivative of x as in (2.1), then it also satisfies the Volterra fractional order integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))ds,\ t_0 \leq t \leq t_0 + T \tag{2.8}$$

and that is every solution of (2.6) is also a solution of (2.8), for detail please see Lakshmikantham et al. (2009).

Let us give the definition of the strict stability criteria for unperturbed fractional differential systems with initial time difference.

Definition 2.1: The solution $y(t, \tau_0, y_0)$ of the fractional order differential system (2.7) through (τ_0, y_0) is said to be initial time difference strict stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the fractional order differential system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$. If given any $\epsilon_1 > 0$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_1 = \delta_1(\epsilon_1, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon_1, \tau_0) > 0$ such that

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| < \epsilon_1 \text{ for } t \geq \tau_0$$

whenever $\|y_0 - x_0\| < \delta_1$ and $|\tau_0 - t_0| < \delta_2$ and, for $\delta_1^* < \delta_1$ and $\delta_2^* < \delta_2$ there exist $\epsilon_2 < \min\{\delta_1^*, \delta_2^*\}$ such that

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| > \epsilon_2 \text{ for } t \geq \tau_0$$

whenever $\|y_0 - x_0\| > \delta_1^*$ and $|\tau_0 - t_0| > \delta_2^*$.

Definition 2.2: If δ_1, δ_2 and ϵ_2 in Definition 2.1 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the system (2.7) is initial time difference uniformly strict stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$.

Definition 2.3: The solution $y(t, \tau_0, y_0)$ of the system (2.7) through (τ_0, y_0) is said to be initial time difference strictly attractive in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$. If given any $\alpha_1 > 0, \gamma_1 > 0, \epsilon_1 > 0$ and $\tau_0 \in \mathbb{R}_+$, for every $\alpha_2 < \alpha_1$ and $\gamma_2 < \gamma_1$, there exist $\epsilon_2 < \epsilon_1, T_1 = T_1(\epsilon_1, \tau_0)$ and $T_2 = T_2(\epsilon_1, \tau_0)$ such that

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| < \epsilon_1 \text{ for } T_1 + \tau_0 \leq t \leq T_2 + \tau_0$$

whenever $\|y_0 - x_0\| < \alpha_1$ and $|\tau_0 - t_0| < \gamma_1$ and

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| > \epsilon_2 \text{ for } T_2 + \tau_0 \geq t \geq T_1 + \tau_0$$

whenever $\|y_0 - x_0\| > \alpha_2$ and $|\tau_0 - t_0| > \gamma_2$.

If $T_1$and T_2 in Definition 2.3 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the system (2.7) is initial time difference uniformly strictly attractive stable with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$.

Definition 2.4: The solution $y(t, \tau_0, y_0)$ of the system (2.7) through (τ_0, y_0) is said to be initial time difference strictly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ if Definition 2.3 satisfies and the solution $y(t, \tau_0, y_0)$ of the system (2.7) through (τ_0, y_0) is initial time difference strictly stable with respect to the solution $x(t-\eta, t_0, x_0)$.

If $T_1$and T_2 in Definition 2.3 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the system (2.7) is initial time difference uniformly strictly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$.

Definition 2.5: For any real-valued function $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, we define the fractional order Dini derivatives in Caputo's sense

$$^cD_+^q V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h^q}[V(t,x) - V(t-h, x - h^q f(t,x))]$$

where $x(t) = x(t, t_0, x_0)$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 2.6: For a real-valued function $V(t,x) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ we define the generalized fractional order derivatives (Dini-like derivatives) in Caputo' s sense ${}_*^cD_+^q V(t, y - \widetilde{x})$ as follows

$${}_*^cD_+^q V(t, y - \widetilde{x})$$

$$= \lim_{h\to 0^+} \sup\left[\frac{V(t, y - \widetilde{x}) - V(t-h, y - \widetilde{x} - h^q(f(t,y) - \widetilde{f}(t, \widetilde{x})))}{h^q}\right]$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 2.7: $\mathcal{K}$ is said to be the class $\mathcal{K}$ set of functions such that

$\mathcal{K} := [a : a \in C([0,\rho], \mathbb{R}_+)$, a is strictly monotone increasing and $a(0) = 0]$.

3. MAIN RESULTS

In this section we obtain the strict stability concepts with initial time difference for fractional differential equations parallel to the Lyapunov's results.

Theorem 3.1: Assume that

(A_1) for each μ , $0 < \mu < \rho, V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $\|z\| \geq \mu$,

$$b_1(\|z\|) \leq V_\mu(t,z) \leq a_1(\|z\|), a_1, b_1 \in \mathcal{K}$$

$$ {}^c_*D^q_+ V_\mu(t,z) \leq 0; \tag{3.1}$$

(A_2) for each $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $\|z\| \leq \theta$,

$$b_2(\|z\|) \leq V_\theta(t,z) \leq a_2(\|z\|), a_2, b_2 \in \mathcal{K}$$

$$ {}^c_*D^q_+ V_\theta(t,z) \geq 0; \tag{3.2}$$

where $z(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ for $t \geq \tau_0, y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) and $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$.

Then the solution $y(t,\tau_0,y_0)$ of the system (2.7) is the initial time difference strictly stable in fractional case with respect to $x(t-\eta,t_0,x_0)$ of the system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$.

Proof of Theorem 3.1: Let us assume that $0 < \epsilon_1 < \rho$ and $\tau_0 \in \mathbb{R}_+$. Let us choose that $\delta_1 = \delta_1(\epsilon_1, \tau_0) > 0$ such that

$$a_1(\delta_1) < b_1(\epsilon_1) \tag{3.3}$$

since we have $b_1(\epsilon_1) \leq a_1(\delta_1)$ in (A_1). Then we claim that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon_1 \text{ for } t \geq \tau_0 \tag{3.4}$$

whenever $\|y_0 - x_0\| < \delta_1$ and $|\tau_0 - t_0| < \delta_2$.

If (3.4) is not true, then there exist $t_1 > t_2 > \tau_0$ and the solution of (2.6) and by using (3.1) with $\|y_0 - x_0\| < \delta_1, |\tau_0 - t_0| < \delta_2$ satisfying

$$\left\|y(t_1) - \widetilde{x}(t_1)\right\| = \epsilon_1, \left\|y(t_2) - \widetilde{x}(t_2)\right\| = \delta_1$$
$$\text{and } \delta_1 \leq \left\|y(t) - \widetilde{x}(t)\right\| \leq \epsilon_1 \text{ for } t \in [t_2, t_1]$$

where $\widetilde{x}(t) = x(t-\eta,t_0,x_0)$.

Let us set $\mu = \delta_1$, we can obtain that

$$\begin{aligned} b_1(\epsilon_1) &= b_1(\left\|y(t_1) - \widetilde{x}(t_1)\right\|) \leq V_\mu(t_1, y(t_1) - \widetilde{x}(t_1)) \\ &\leq V_\mu(t_2, y(t_2) - \widetilde{x}(t_2)) \\ &\leq a_1(\left\|y(t_2) - \widetilde{x}(t_2)\right\|) = a_1(\delta_1) \\ b_1(\epsilon_1) &\leq a_1(\delta_1) \end{aligned}$$

which contradicts with (3.3). Hence, (3.4) is valid.

Now let $0 < \delta_1^* < \delta_1, 0 < \delta_2^* < \delta_2$ and $\epsilon_2 < \delta = \min\{\delta_1^*, \delta_2^*\}$ such that

$$a_2(\epsilon_2) < b_2(\delta). \tag{3.5}$$

since we have $a_2(\epsilon_2) \geq b_2(\delta)$ in (A_2). Then we can prove that

$$\epsilon_2 < \|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon_1 \text{ for } t \geq \tau_0 \tag{3.6}$$

whenever $\delta_1^* < \|y_0 - x_0\| < \delta_1$ and $\delta_2^* < |\tau_0 - t_0| < \delta_2$.

In fact, if (3.6) is not true, then there would exist $t_1 > t_2 > \tau_0$ and the solution of (2.6) and by using (3.2) with $\delta_1^* < \|y_0 - x_0\| < \delta_1, \delta_2^* < |\tau_0 - t_0| < \delta_2$ satisfying

$$\left\|y(t_1) - \widetilde{x}(t_1)\right\| = \epsilon_2, \left\|y(t_2) - \widetilde{x}(t_2)\right\| = \delta$$
$$\text{and } \left\|y(t) - \widetilde{x}(t)\right\| \leq \delta \text{ for } t \in [t_2, t_1]. \tag{3.7}$$

Let us set $\theta = \delta$ and by using (A_2), we get

$$\begin{aligned} a_2(\epsilon_2) &= a_2(\left\|y(t_1) - \widetilde{x}(t_1)\right\|) \geq V_\theta(t_1, y(t_1) - \widetilde{x}(t_1)) \\ &\geq V_\theta(t_2, y(t_2) - \widetilde{x}(t_2)) \\ &\geq b_2(\left\|y(t_2) - \widetilde{x}(t_2)\right\|) = b_2(\delta) \\ a_2(\epsilon_2) &\geq b_2(\delta) \end{aligned}$$

which contradicts with (3.5). Thus (3.6) is valid. Then the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) is initial time difference strictly stable in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$ for $t \geq \tau_0$.

This completes the proof of Theorem 3.1.

If δ_1, δ_2 and ϵ_2 is independent of τ_0, then the solution $y(t,\tau_0,y_0)$ of the system (2.7) is initial time difference uniformly strict stable in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$ for $t \geq \tau_0$.

Theorem 3.2: Assume that

(A_1) for each μ , $0 < \mu < \rho, V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $\|z\| \geq \mu$,

$$b_1(\|z\|) \leq V_\mu(t,z) \leq a_1(\|z\|), a_1, b_1 \in \mathcal{K},$$

$$ {}^c_*D^q_+ V_\mu(t,z) \leq -c_1(\|z\|), \; c_1 \in \mathcal{K}; \tag{3.8}$$

(A_2) for each $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $\|z\| \leq \theta$,

$$b_2(\|z\|) \leq V_\theta(t,z) \leq a_2(\|z\|), a_2, b_2 \in \mathcal{K},$$

$${}^{c}_{*}D^{q}_{+}V_\theta(t,z) \geq -c_2(\|z\|) \; c_2 \in \mathcal{K}; \qquad (3.9)$$

where $z(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ for $t \geq \tau_0, y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) and $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$.

Then the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) is the initial time difference uniformly strictly asymptotically stable in fractional case with respect to $x(t-\eta,t_0,x_0)$ of the solution of the system (2.6) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$.

Proof of Theorem 3.2: We note that (3.8) implies (3.1). However, (3.9) does not yield (3.2). As a result of these, we obtain because of (3.8) only uniformly stability of unperturbed systems with initial time difference with respect to $x(t-\eta,t_0,x_0)$ that is for given any $\epsilon_1 \leq \rho$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_{10} = \delta_{10}(\epsilon_1) > 0$ and $\delta_{20} = \delta_{20}(\epsilon_1) > 0$ such that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon_1 \text{ for } t \geq \tau_0 \; .$$

$$\text{whenever } \|y_0 - x_0\| < \delta_{10} \text{and} \, |\tau_0 - t_0| < \delta_{20}$$

To prove the conclusion of Theorem 3.2 we need to show that the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) for $t \geq \tau_0$ is strictly uniformly attractive in fractional case with respect to $x(t-\eta,t_0,x_0)$ for this purpose and for $t \geq \tau_0$, let $\epsilon_1 = \rho$ and set $\delta_{10} = \delta_1(\rho)$ and $\delta_{20} = \delta_2(\rho)$ so that (3.10) yields $\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \rho$ for $t \geq \tau_0$ whenever $\|y_0 - x_0\| < \delta_1$ and $|\tau_0 - t_0| < \delta_2$.

Let $\|y_0 - x_0\| < \delta_{10}$ and $|\tau_0 - t_0| < \delta_{20}$. We show, using standard argument, that there exists a $t^* \in [\tau_0, \tau_0 + T]$, we choose $T = T(\epsilon,\tau_0) \geq \left(\frac{a_1(\max\{\delta_{10},\delta_{20}\})}{c_1(\min\{\delta_1,\delta_2\})}\Gamma(q+1)\right)^{\frac{1}{q}}$ where δ_{10} and δ_{20} are the numbers corresponding to ϵ_1 in (3.1.10) that is in stability of unperturbed systems with initial time difference with respect to $x(t-\eta,t_0,x_0)$ such that $\|y(t^*,\tau_0,y_0) - x(t^*-\eta,t_0,x_0)\| < \delta_1$, $t^* \geq \tau_0$ for any solutions of the systems (2.1.1) and (2.1.3) with $\|y_0 - x_0\| < \delta_{10}$ and $|\tau_0 - t_0| < \delta_{20}$. If this is not true, we will have $\|y(t^*,\tau_0,y_0) - x(t^*-\eta,t_0,x_0)\| \geq \delta_1$ for $t^* \in [\tau_0, \tau_0 + T]$. Then, $\mu = \delta_1$ and using (A_1) with (3.1.8), we have in view of the choice of T,

$$0 < b_1(\delta_1) \leq b_1(\left\|y(\tau_0+T) - \widetilde{x}(\tau_0+T)\right\|)$$
$$\leq V_\mu(\tau_0+T, y(\tau_0+T) - \widetilde{x}(\tau_0+T))$$

$$\leq V_\mu(\tau_0, y_0 - x_0)$$

$$-\frac{1}{\Gamma(q)}\int_{\tau_0}^{\tau_0+T}(t-s)^{q-1}c_1(\left\|y(s) - \widetilde{x}(s)\right\|)ds$$

$$\leq a_1(\max\{\delta_{10},\delta_{20}\}) - \frac{c_1(\min\{\delta_1,\delta_2\})}{\Gamma(q)}\int_{\tau_0}^{\tau_0+T}(t-s)^{q-1}ds$$

$$\leq a_1(\max\{\delta_{10},\delta_{20}\}) - \frac{c_1(\min\{\delta_1,\delta_2\})}{\Gamma(q+1)}T^q$$

$$\leq 0$$

This contradiction implies that there exist a $t^* \in [\tau_0, \tau_0 + T]$ satisfying $\|y(t^*,\tau_0,y_0) - x(t^*-\eta,t_0,x_0)\| < \delta_1$ for $t^* \geq \tau_0$. Because of the uniform stability $y(t,\tau_0,y_0)$ of (2.7) with initial time difference with respect to $x(t-\eta,t_0,x_0)$ related to the solution of (2.6), this yields that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon_1 \text{ for } t \geq \tau_0 + T \geq t^*$$

which implies that there exists a $\tau_0 < T_1 < T$ such that

$$\|y(\tau_0+T,\tau_0,y_0) - x(\tau_0+T-\eta,t_0,x_0)\| = \epsilon_1.$$

Now, for any $\delta_{12}, 0 < \delta_{12} < \delta_{10}$ and $0 < \delta_{12} < \delta_{20}$ we can choose ϵ_2 such that $b_2(\epsilon_1) > a_2(\epsilon_2)$ and $0 < \epsilon_2 < \epsilon_1 < \delta_{12}$.

Suppose that $\delta_{12} < \|y_0 - x_0\| < \min\{\delta_{10},\delta_{20}\}$ and $\delta_{12} < |\tau_0 - t_0| < \min\{\delta_{10},\delta_{20}\}$. Let us define $\tau = [\frac{\Gamma(q)(b_2(\epsilon_1) - a_2(\epsilon_2))}{c_2(\epsilon_1)}]^{\frac{1}{q}}$, and $T_2 = T_1 + \tau$.

Since, $\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \leq \epsilon_1$ for $t \geq \tau_0 + T_1$, choosing $\theta = \epsilon_1$ and using (A_2) with (3.9) we have for $t \in [\tau_0 + T_1, \tau_0 + T_2]$,

$$a_2(\left\|y(t) - \widetilde{x}(t)\right\|) \geq V_\theta(t, y(t) - \widetilde{x}(t))$$
$$\geq V_\theta(\tau_0+T_1, y(\tau_0+T_1) - \widetilde{x}(\tau_0+T_1))$$

$$-\frac{1}{\Gamma(q)}\int_{\tau_0+T_1}^{t}(t-s)^{q-1}c_2(\left\|y(s) - \widetilde{x}(s)\right\|)ds$$
$$\geq b_2(\epsilon_1)$$

$$-\frac{1}{\Gamma(q)}\int_{\tau_0+T_1}^{t}(t-s)^{q-1}c_2(\left\|y(s) - \widetilde{x}(s)\right\|)ds$$
$$\geq b_2(\epsilon_1) - \frac{c_2(\epsilon_1)}{\Gamma(q)}[t - (\tau_0+T_1)]^q$$

Since, $t - (\tau_0 + T_1) > \tau$ and a_2^{-1} exists, it follows that

$$a_2(\left\|y(t) - \widetilde{x}(t)\right\|) >$$
$$b_2(\epsilon_1) - \frac{c_2(\epsilon_1)}{\Gamma(q)}[\frac{\Gamma(q)(b_2(\epsilon_1) - a_2(\epsilon_2))}{c_2(\epsilon_1)}] = a_2(\epsilon_2).$$

This yields that

$\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|>\epsilon_2$ for $t\in[\tau_0+T_1,\tau_0+T_2]$ and therefore,

$$\epsilon_2<\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|<\epsilon_1$$

for $t\in[\tau_0+T_1,\tau_0+T_2]$.

This completes the proof. Then the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) is initial time difference uniformly strictly asymptotically stable in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.6) for $t\geq\tau_0\geq 0, t_0\in\mathbb{R}_+$ and $\eta=\tau_0-t_0$.

Before we prove the general result in terms of the comparison principle. Let us consider the uncoupled comparison fractional differential systems in Caputo's sense:

$$\begin{cases}(i)\,{}^cD^q\,u_1=g_1(t,u_1),u_1(\tau_0)=u_{10}\geq 0\\(ii)\ {}^cD^qu_2=g_2(t,u_2),u_2(\tau_0)=u_{20}\geq 0\end{cases}\tag{3.11}$$

where $g_1,g_2\in C\left[\mathbb{R}_+^2,\mathbb{R}\right]$ The comparison system (3.11) is said to be strictly stable in fractional case:

If given any $\epsilon_1>0$ and $t\geq\tau_0,\tau_0\in\mathbb{R}_+$, there exist a $\delta_1>0$ such that

$$u_{10}\leq\delta_1 \text{ implies } u_1(t)<\epsilon_1 \text{ for } t\geq\tau_0$$

and for every $\delta_2<\delta_1$ there exists an $\epsilon_2>0, 0<\epsilon_2<\delta_2$ such that

$$u_{20}\geq\delta_2 \text{ implies } u_2(t)>\epsilon_2 \text{ for } t\geq\tau_0.$$

Here, $u_1(t)$ and $u_2(t)$ are any solutions of (i) in (3.11) and (ii) in (3.11); respectively.

The comparison system (3.11) is said to be strictly attractive in fractional case:

If given any $\alpha_1>0,\gamma_1>0,\epsilon_1>0$ and $\tau_0\in\mathbb{R}_+$, for every $\alpha_2<\alpha_1$, there exist $\epsilon_2<\epsilon_1, T_1=T_1(\epsilon_1,\tau_0)>0$ and $T_2=T_2(\epsilon_1,\tau_0)>0$ such that

$$u_1(t,\tau_0,u_0)<\epsilon_1 \text{ for } T_1+\tau_0\leq t\leq T_2+\tau_0 \text{ when } u_{10}\leq\alpha_1$$

and

$$u_2(t,\tau_0,u_0)>\epsilon_2 \text{ for } T_2+\tau_0\geq t\geq T_1+\tau_0 \text{ when } u_{20}\geq\alpha_2.$$

If $T_1$and T_2 are independent of τ_0, then the comparison system (3.11) is initial time difference uniformly strictly attractive in fractional case for $t\geq\tau_0$.

Following main result based on this definition that result is formulated in terms of comparison principle.

Theorem 3.3: Assume that

(A_1) for each μ , $0<\mu<\rho, V_\mu\in C[\mathbb{R}_+\times S_\rho,\mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t,z)\in\mathbb{R}_+\times S_\rho$ and $\|z\|\geq\mu$,

$$b_1(\|z\|)\leq V_\mu(t,z)\leq a_1(\|z\|), a_1,b_1\in\mathcal{K},$$

$${}^c_*D^q_+V_\mu(t,z)\leq g_1(t,V_\mu(t,z));\tag{3.12}$$

(A_2) for each $\theta, 0<\theta<\rho, V_\theta\in C[\mathbb{R}_+\times S_\rho,\mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t,z)\in\mathbb{R}_+\times S_\rho$ and $\|z\|\leq\theta$,

$$b_2(\|z\|)\leq V_\theta(t,z)\leq a_2(\|z\|), a_2,b_2\in\mathcal{K},$$

$${}^c_*D^q_+V_\theta(t,z)\geq g_2(t,V_\theta(t,z));\tag{3.13}$$

where $g_2(t,u)\leq g_1(t,u), g_1,g_2\in C[\mathbb{R}^2_+,\mathbb{R}], g_1(t,0)=g_2(t,0)=0$ and $z(t)=y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)$ for $t\geq\tau_0, y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) and $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.6) for $t\geq\tau_0\geq 0, t_0\in\mathbb{R}_+$ and $\eta=\tau_0-t_0$.

Then any strict stability concept in fractional case of the comparison system implies the corresponding strict stability concept in fractional case of the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) with respect to the solution $x(t-\eta,t_0,x_0)$ of the system (2.6) with initial time difference where $x(t,t_0,x_0)$ is any solution of the system (2.6) for $t\geq\tau_0\geq 0,\ t_0\in\mathbb{R}_+$.

Proof of Theorem 3.3: We will only prove the case of strict uniformly asymptotically stability in fractional case. Suppose that the comparison fractional differential systems in (3.11) is strictly uniformly asymptotically stable in fractional case, then for any given $\epsilon_1,\ 0<\epsilon_1<\delta$, there exist a $\delta^*>0$ such that $u_{10}\leq\delta^*$ implies that $u_1(t,\tau_0,u_{10})<b_1(\epsilon_1)$ for $t\geq\tau_0$.

For this ϵ_1 we choose δ_1 and δ_{11}, such that $a_1(\delta_1^*)\leq\delta^*$ and $\delta_1^*<\epsilon_1$ where $\delta_1^*=\max\{\delta_1,\delta_{11}\}$, then we claim that

$$\|y_0-x_0\|<\delta_1, |\tau_0-t_0|<\delta_{11}$$

$$\text{imply that } \|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|<\epsilon_1 \text{ for } t\geq\tau_0.\tag{3.14}$$

If it is not true, then there exist t_1 and $t_2,\ t_2>t_1>\tau_0$ and a solution $z(t)$ of

$${}^cD^qz=\widetilde{f}(t,z),\ z(\tau_0)=y_0-x_0 \text{ for } t\geq\tau_0$$

with $|\tau_0-t_0|<\delta_{11}$ and $\|y_0-x_0\|<\delta_1$

$$\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|<\delta_1^*,$$
$$\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|=\epsilon_1 \text{ and}$$

$$\delta_1^*\leq\|y(t,\tau_0,y_0)-x(t-\eta,t_0,x_0)\|<\epsilon_1 \text{ for } [t_1,t_2].$$

Choosing $\mu=\delta_1^*$ and using the theory of differential inequalities we get

$$
\begin{aligned}
b_1(\epsilon_1) &= b_1(\|y(t_2,\tau_0,y_0) - x(t_2-\eta,t_0,x_0)\|) \\
&\leq V_\mu(t_2, y(t_2,\tau_0,y_0) - x(t_2-\eta,t_0,x_0)) \\
&\leq r(t_2,t_1,V_\mu(t_1,y(t_1,\tau_0,y_0) - x(t_1-\eta,t_0,x_0))) \\
&\leq r(t_2,t_1,a_1(\delta_1^*)) \\
&\leq r(t_2,t_1,\delta^*) \\
&< b_1(\epsilon_1). \\
b_1(\epsilon_1) &< b_1(\epsilon_1).
\end{aligned}
$$

which is a contradiction. Here $r(t,\tau_0,u_{10})$ is the maximal solution of (3.11). Hence, (3.14) is true and we have uniformly stability in fractional case with initial time difference. Now, we shall prove strictly uniformly attractive in fractional case with initial time difference.

For any given $\delta_2,\ \epsilon_2 > 0,\ \delta_2 < \delta^*$, we choose $\bar{\delta}_2$ and $\bar{\epsilon}_2$ such that $a_1(\delta_2) < \bar{\delta}_2$ and $b_1(\epsilon_2) \geq \bar{\epsilon}_2$. For these $\bar{\delta}_2$ and $\bar{\epsilon}_2$, since (3.11) is strictly uniformly attractive in fractional case, for any $\bar{\delta}_3 < \bar{\delta}_2$ there exist $\bar{\epsilon}_3$ and T_1 and T_2 (we assume $T_2 < T_1$) such that $\bar{\delta}_3 < u_{10} = u_{20} < \bar{\delta}_2$ implies

$$
\begin{aligned}
r(t,\tau_0,u_{10}) &\leq r(t,\tau_0,\bar{\delta}_2) < \bar{\epsilon}_2 \\
\rho(t,\tau_0,u_{20}) &\geq \rho(t,\tau_0,\bar{\delta}_3) > \bar{\epsilon}_2
\end{aligned}
$$

where $r(t,\tau_0,u_{10})$ and $\rho(t,\tau_0,u_{20})$ is the maximal solution and minimal solution of (3.11) (i) and (3.11) (ii); respectively.

Now, for any δ_3, let $b_2(\delta_3) \geq \bar{\delta}_3$. We choose ϵ_3 such that $a_2(\epsilon_3) < \bar{\epsilon}_3$. Then by using comparison principle (3.11) (i) and (A_1), we have

$$
\begin{aligned}
b_1(\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\|) &\leq V_\mu(t, y(t,\tau_0,y_0) \\
&\quad -x(t-\eta,t_0,x_0)) \\
&\leq r(t,\tau_0,V_\mu(\tau_0,y_0-x_0)) \\
&\leq r(t,\tau_0,a_1(\|y_0-x_0\|)) \\
&\leq r(t,\tau_0,\bar{\delta}_2) \\
&< \bar{\epsilon}_2 \leq b_1(\epsilon_2) \\
b_1(\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\|) &< b_1(\epsilon_2)
\end{aligned}
$$

since b_1^{-1} exists which implies that
$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \epsilon_2$ for $t \in [\tau_0+T_2, \tau_0+T_1]$.

Similarly, by using comparison principle in (3.11) (ii) and (A_2) we get

$$
\begin{aligned}
a_2(\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\|) &\geq V_\theta(t, y(t,\tau_0,y_0) \\
&\quad -x(t-\eta,t_0,x_0)) \\
&\geq \rho(t,\tau_0,V_\theta(\tau_0,y_0-x_0)) \\
&\geq \rho(t,\tau_0,b_2(\delta_3)) \\
&\geq \rho(t,\tau_0,\bar{\delta}_3) \\
&> \bar{\epsilon}_3 \geq a_2(\epsilon_3) \\
a_2(\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\|) &> a_2(\epsilon_3)
\end{aligned}
$$

since a_2^{-1} exists which implies that for
$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| > \epsilon_3$ for $t \in [\tau_0+T_2, \tau_0+T_1]$. Hence, the solution $y(t,\tau_0,y_0)$ of the system (2.7) through (τ_0,y_0) is strictly uniformly attractive in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$ is any solution of the system (2.6) for $t \geq \tau_0 \geq 0,\ t_0 \in \mathbb{R}_+$. The proof is completed.

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